Systems of Linear Equations Linear Algebra X. Du

- Example: suppose we are given the following system of three linear equations with three unknowns:
  - $$a_1x + b_1y + c_1z = d_1$$
    $$a_2x + b_2y = c_2z = d_2$$
    $$a_3x + b_3y = c_3z = d_3$$
- Vector equation form:
  - Shows the system as a linear combination of vectors.
  - Example: $x\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + y\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + z\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$
- Matrix equation form:
  - Takes the form $A\vec{x} = \vec{b}$, where A is a matrix
  - Example: $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$
  - $\vec{x} = A^{-1}\vec{b}$
    - Rearrange $A\vec{x} = \vec{b}$ to get this solution
    - $A\vec{x} = \vec{b}$ has a unique solution iff A is invertible (iff $\det(A) \neq 0$).
- Associated augmented matrix:
  - $\begin{pmatrix} A & \vec{b} \end{pmatrix}$ where $A\vec{x} = \vec{b}$ is the matrix equation representation of the system
  - Example: $\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$
  - If the solution to the system is unique, then the last column vector of the reduced row-echelon form of the associated augmented matrix is that unique solution.
- **Underdetermined**: less equations than unknowns
- **Overdetermined**: more equations than unknowns
- **Homogeneous system of linear equations**
  - Matrix equation takes the form $A\vec{x} = \vec{0}$.
  - The last column of the associated augmented matrix is $\vec{0}$.
  - $\vec{0}$ is always a solution. Hence, this is called the **trivial solution**.
    - Solutions not $\vec{0}$ are non-trivial solutions
  - Any linear combination of solutions is also a solution.
  - Two cases:
    - 1. System has only the trivial solution (iff A is invertible).
    - 2. System has infinitely many solutions. This is because any linear combination of solutions is also a solution.